The Astrophysical Journal, 566:L41–L44, 2002 February 10

EFFECT OF FRACTIONAL KINETIC HELICITY ON TURBULENT MAGNETIC DYNAMO SPECTRA

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Received 2001 November 29; accepted 2002 January 10; published 2002 January 21

ABSTRACT

Magnetic field amplification in astrophysics ultimately requires an understanding of MHD turbulence. Kinetic helicity has long been known to be important for large-scale field growth in forced MHD turbulence and has been recently demonstrated numerically to be asymptotically consistent with slow mean field dynamo action in a periodic box. Here we show numerically that the magnetic spectrum at and below the forcing scale is also strongly influenced by kinetic helicity. We identify a critical value, $f_{h,\,\mathrm{crit}}$, above which the magnetic spectrum develops maxima at a wavenumber of 1 scale *and* at the forcing scale. For $f < f_{h,\,\mathrm{crit}}$, the field peaks only at the resistive scale. Kinetic helicity may thus be important not only for generating a large-scale field, but also for establishing observed peaks in magnetic spectra at the forcing scale. The turbulent Galactic disk provides an example where both large-scale (greater than the supernova forcing scale) fields and small-scale (less than or equal to forcing scale, with peak at forcing scale) fields are observed. We discuss this and the potential application to the protogalaxy, but we also emphasize the limitations in applying our results to these systems.

Subject headings: galaxies: magnetic fields — ISM: magnetic fields — methods: numerical — MHD — stars: magnetic fields — turbulence

1. INTRODUCTION

The origin of magnetic fields and the dynamics of three-dimensional magnetohydrodynamic (MHD) turbulence in astrophysical sources are problems of long-standing interest (e.g., Cowling 1957; Moffatt 1978; Parker 1979; Krause & Rädler 1980; Zeldovich, Ruzmaikin, & Sokoloff 1984). The standard in situ mean field dynamo (MFD) model (Moffatt 1978; Parker 1979; Krause & Rädler 1980) of the large-scale (i.e., scales greater than the turbulent forcing) magnetic field origin can be thought of as a framework for understanding an inverse cascade of magnetic helicity, initiated by a forcing of kinetic helicity (Pouquet, Frisch, & Leorat 1975). While the role of kinetic and magnetic helicities are important for in situ nonlocal inverse cascade models of large-scale fields, or MFDs, the small-scale (i.e., scales at or below the turbulent forcing scale) dynamo does not explicitly require helicity to amplify the total magnetic energy density (Zeldovich et al. 1983; Parker 1979). Nonhelical forced turbulent amplification of the small-scale fields and fully helical forced growth of large-scale fields have been recently simulated (Cho & Vishniac 2001; Chou 2001; Brandenburg 2001; Maron & Cowley 2001).

But there is an important subtlety that has not yet been addressed. Although numerical work generically shows that the total energy of the small-scale field in turbulent media saturates at nearly equipartition with the kinetic energy spectrum, nonhelical small-scale dynamos produce a peak of the magnetic energy spectrum on the resistive scale for magnetic Prandtl number of ≥1, not on the forcing scale (Chou 2001; Maron & Cowley 2001). This contradicts, for example, observations of the Galactic magnetic field, which has a peak in the spectrum on the turbulent forcing scale (Beck et al. 1996), and maximally helical simulations (Brandenburg 2001). Here we show that forcing with varying levels of fractional kinetic helicity affects the overall spectral shape at large *and* small scales.

In § 2 we discuss the equations and the simulations. In § 3 we give the results, and in § 4 the interpretations. We conclude in § 5.

2. EQUATIONS AND NUMERICAL SCHEME

We investigate forced helical MHD turbulence. We write the magnetic field in velocity units and so define $\boldsymbol{b} \equiv \boldsymbol{B}/\sqrt{4\pi}$, where $\boldsymbol{B}$ is the magnetic field. Incompressibility is assumed throughout, so we set density to $\rho = 1$, and $\nabla \cdot \boldsymbol{v} = 0$, where $\boldsymbol{v}$ is velocity. We include the thermal pressure P and the magnetic pressure in the total pressure $p = P + b^2/2$ and assume isotropic kinetic and magnetic viscosities ν_v. The MHD equations become

$$\partial_t \boldsymbol{v} = -\boldsymbol{v} \cdot \nabla \boldsymbol{v} - \nabla p + \boldsymbol{b} \cdot \nabla \boldsymbol{b} + \nu_v \nabla^2 \boldsymbol{v}, \tag{1}$$

$$\partial_t \boldsymbol{b} = -\boldsymbol{v} \cdot \nabla \boldsymbol{b} + \boldsymbol{b} \cdot \nabla \boldsymbol{v} + \nu_b \nabla^2 \boldsymbol{b}. \tag{2}$$

To relate p to $\boldsymbol{v}$ and $\boldsymbol{b}$, we take the divergence of equation (1), which upon inversion, yields

$$p = \int \frac{d^3x'}{4\pi} \frac{(\nabla \boldsymbol{v} : \nabla \boldsymbol{v} - \nabla \boldsymbol{b} : \nabla \boldsymbol{b})}{|\boldsymbol{x}' - \boldsymbol{x}|}. \tag{3}$$

A random forcing field with energy $\epsilon_f \cdot \Delta t$ is generated at each time step and is added to the existing velocity field, where ϵ_f is the average forcing power. The amplitudes of the forcing-field Fourier modes are assigned according to a specified power spectrum, with the energy selected from a Boltzmann distribution. The mode phases are random within the constraint of divergencelessness. We input kinetic helicity $\boldsymbol{v} \cdot \nabla \times \boldsymbol{v}$ at the forcing wavenumber of $s = k/2\pi = 4.5$ by making a randomly determined subset of the Fourier modes maximally helical, leaving the rest unchanged. The fraction of maximally helical modes is f_h, which we denote "fractional helicity." In contrast, simulations of Maron & Cowley (2001) invoked zero mean magnetic field and zero mean kinetic helicity. Only fractional random fluctuations of the kinetic helicity on the order of 10%

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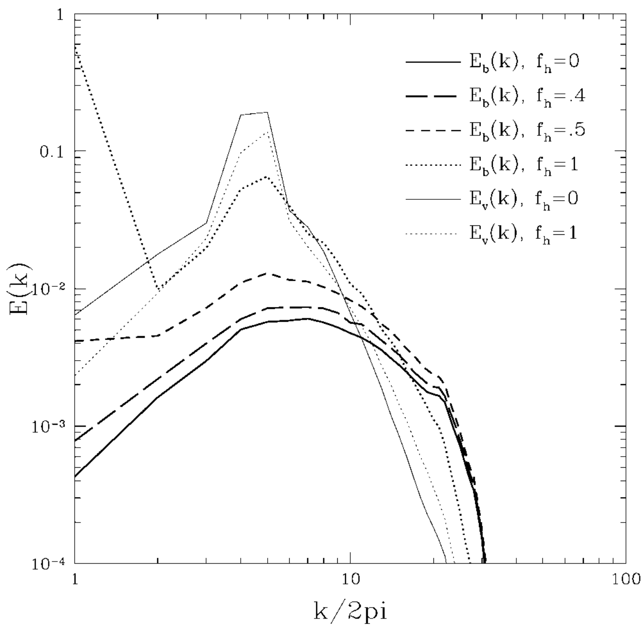

FIG. 1.—Saturated kinetic and magnetic energy spectra for successive values of f_h. Simulations are A0, A4, A5, and A10. The kinetic spectra are identical for $f_h \le 0.4$

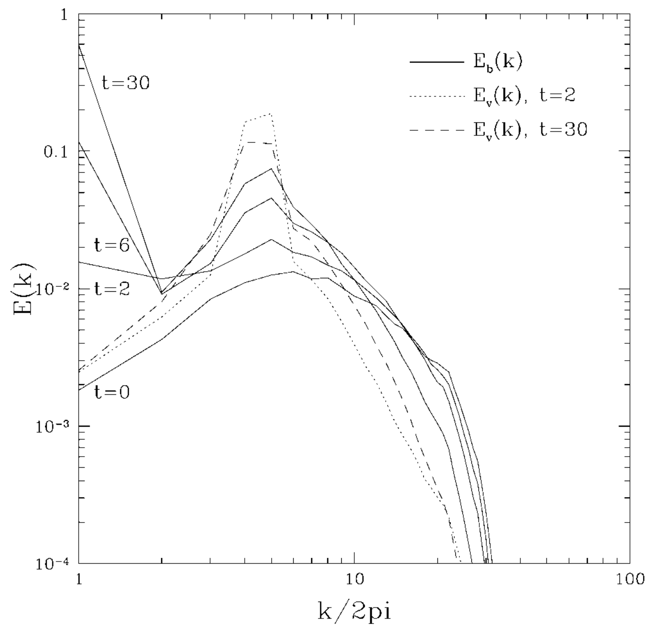

FIG. 2.—Time sequence of kinetic and magnetic energy spectra for simulation A10.

were present. The magnetic helicity was also initially zero and subsequently fluctuated about zero at an amplitude of 10% of the maximum. The equations of MHD are solved spectrally. The turbulence is incompressible, and the boundaries are periodic. (Including compressibility, while ultimately important for detailed applications, is not expected to have a dramatic effect on the qualitative conclusions herein regarding the role of helicity.) Wavenumbers and physical scales are related by $\lambda k = 2\pi$. Viscosity and resistivity are of the k^2 type ($\nu_v \nabla^2 v$ and $\nu_b \nabla^2 b$). The code is exhaustively discussed in Maron & Goldreich (2001). We note that at each time step, the time derivative of the magnetic helicity is equal to the volume integral of the current helicity times the resistivity. The helicity conservation equation is satisfied and this is important in what follows.

The other key parameters are as follows: the magnetic Prandtl number is $\mathrm{Pr} = \nu_v/\nu_b \sim \lambda_{\nu_v}^2/\lambda_{\nu_b}^2$, which is the ratio of the viscosity to magnetic diffusivity, where λ_{ν_v} and λ_{ν_b} are the viscous and resistive scales, respectively. We denote v_λ and b_λ as the speed and magnetic field at scale λ and v_f and λ_f as the forcing-scale rms velocity and forcing scale, respectively. When $\mathrm{Pr} \ge 1$, the scales have the ordering $\lambda_f > \lambda_{\nu_v} \ge \lambda_{\nu_b}$.

3. RESULTS

We show results here for a selection of 64^3 simulations, which is sufficient to identify the basic effects of fractional helicity on the location of energy peaks. The f_h ranges from 0 to 1 by increments of 0.1 for simulations A0 through A10. For all A0–A10, we used a 64^3 grid, $s_f = 4.5$, $\nu_v = 3 \times 10^{-3}$, $\nu_b = 1 \times 10^{-3}$, and $\mathrm{Pr} = 3$.

The usual kinetic and magnetic energy spectra are defined as the quantities inside the energy integrals $E_v = \int E_v(s)ds$ and $E_b = \int E_b(s)ds$, respectively. The spectra for a range of values of f_h are shown in Figure 1. The time evolution of the $f_h = 1$ case is shown in Figure 2, and the time growth of magnetic helicity is shown in Figure 3. Note in these figures that for $f_h \gtrsim f_{h,\mathrm{crit}} \sim 0.5$, the peak at the forcing scale grows, as does the large-scale field. For $f_h < f_{h,\mathrm{crit}}$, the large-scale field decays, no peak appears at the forcing scale, and the magnetic helicity in the box grows very weakly, if at all. Although we present only $\mathrm{Pr} = 3$ cases in the figures, we also performed simulations with $\mathrm{Pr} = 9$ and found that $f_{h,\mathrm{crit}} \sim 0.7$. Thus, we find that $f_{h,\mathrm{crit}}$ increases with Pr.

We checked for hysteresis by using the saturated state of the $f_h = 1$ simulation of Figure 3 as the initial condition for another simulation with $f_h = 0.4$. We found that the magnetic helicity subsequently decayed to the same value as in the simulation

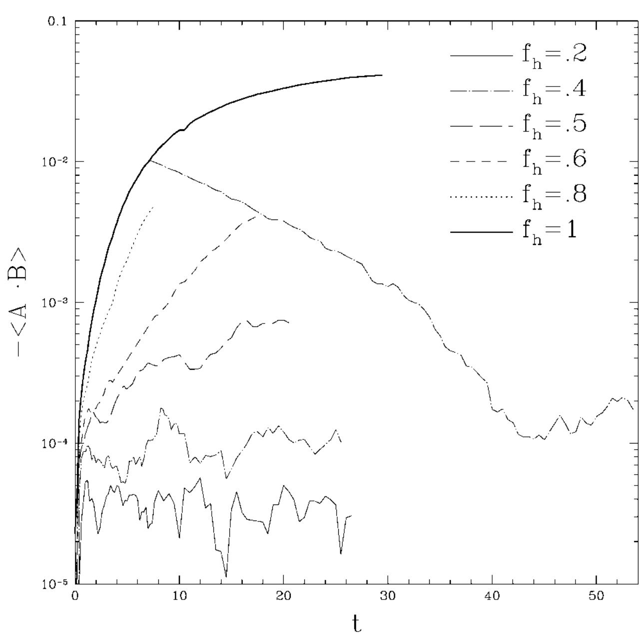

FIG. 3.—Evolution of the magnetic helicity as a function of f_h for simulations from series A. At $t = 7$, simulation A10 ($f_h = 1$) has developed a large-scale magnetic field.

that started with a weak mean field with initial $f_h = 0.4$. There was no evidence for hysteresis.

4. DISCUSSION

Forced helical turbulence was studied systematically in Pouquet et al. (1976) using the eddy-damped quasi-normal Markovian closure scheme (Orsag 1970). Forcing with kinetic helicity leads to segregation of magnetic helicity because the magnetic helicity growth equation has a source term that depends on the kinetic helicity. The kinetic helicity inputs one sign of magnetic helicity at small scales, but the opposite sign is generated on large scales. The growth of the large-scale field represents a nonlocal inverse cascade of the magnetic helicity from the forcing scale and can be interpreted as an α^2 MFD (Pouquet et al. 1976; Brandenburg 2001; Blackman & Field 2001; Field & Blackman 2002; also Verma 2001). Averaged over a periodic box, the time evolution of the total magnetic helicity satisfies $\partial_t\langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle = -2\nu_b\langle \boldsymbol{J}\cdot\boldsymbol{B}\rangle$, where $\boldsymbol{A}$ is the vector potential and $\boldsymbol{J}$ is the current density. If we divide $\langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle$ into small-scale and large-scale contributions, we can see that after the large-scale helical field energy grows to $k_s B_s^2/k_l$, where s and l refer to the dominant small and large scales, the large-scale magnetic helicity dominates (Fig. 3). The growth saturates as long as there remains a net current helicity, but the growth rate is resistively limited, implying a "slow" (decreases with increasing k_{ν_b}) MFD. The large-scale field growth of Figure 3 for all $f_h \geq f_{h,\,\mathrm{crit}}$ is slow in this sense. (Note, however, that there is a short initial "fast" phase. Field & Blackman 2002 and Blackman 2002, discuss this in the context of a dynamical quenching model, which fits Brandenburg 2001 and the results therein and the asymptotic quenching of Gruzinov & Diamond 1994 and Bhattacharjee & Yuan 1995.)

The sign of the magnetic helicity of the growing large-scale field is opposite to that of the kinetic helicity. This is consistent with MFD theory if the kinetic helicity dominates the α effect of the MFD: A positive kinetic helicity means that α would be negative. But the growth of the magnetic helicity associated with the large-scale field is proportional to αB_l^2 (Brandenburg 2001; Blackman & Field 2001; Field & Blackman 2002) so that a positive-input kinetic helicity that gives a negative α produces a negative large-scale magnetic helicity. Figure 3 shows that $\langle \boldsymbol{A}\cdot\boldsymbol{B}\rangle$ is dominated by the large-scale contribution.

That the large-scale field growth proceeds asymptotically slowly, would seem to threaten the relevance of, e.g., the Galaxy, as it is commonly argued that the MFD of the Galaxy has to be "fast" (see Ruzmaikin, Shukurov, & Sokoloff 1988; Zweibel & Heiles 1997). Unlike a periodic box, real astrophysical rotators have boundaries and shear and have helicity driven by the combination of the underlying rotation and stratification with a spatial variation in transport coefficients. So these differences will ultimately need to be studied before results from periodic box solutions like ours can be directly applied to astrophysical large-scale field growth (Blackman & Field 2000). Also, the large-scale field for $f_h > f_{h,\,\mathrm{crit}}$ in the simulation becomes superequipartition, as seen in Figures 1 and 2, because it is nearly force free. That being said, the enterprise of periodic box simulations is very worthwhile; even comparing the difference between periodic boxes with more realistic simulations will be important to our further understanding of MHD turbulence, and it allows the demonstration of some principles that emerge in the simplest possible forced nonlinear dynamo system.

In spite of these caveats, the results of the present simulations are provocative: the same $f_{h,\,\mathrm{crit}}$ determines both whether the large-scale field grows *and* whether a peak grows at the forcing scale. The large-scale field growth and the presence of the small-scale peak at the forcing scale appear to be intimately related. Magnetic helicity undoubtedly also plays a role in the drain of the small-scale peak from the resistive scales for $f_h > f_{h,\,\mathrm{crit}}$. Consider the role of magnetic helicity conservation in mode interactions via a slightly more general version of the argument of (Frisch et al. 1975): Suppose an initial state of maximal magnetic helicity is confined to wavenumbers k_m and k_n, with $k_m < k_n$, and suppose the magnetic field dominates the energy at these wavenumbers. We then have $E_b(k_m) + E_b(k_n) = E_T(k_p)$ and $H_b(k_m) + H_b(k_n) = E_b(k_m)/k_m + E_b(k_n)/k_n \leq E_T(k_p)/k_p$, where E_T means the total energy. The last inequality can be satisfied only if $k_p < k_n$. This argument applies only when the scales k_m and k_n are magnetically dominated, although k_p need not be (a point not addressed in Frisch et al. 1975). An initial state for which the field is dominant and therefore satisfies the validity criterion is the small-scale saturated state shown in Figure 1 for $f_h = 0$. We can reason that when sufficient magnetic helicity is imposed, it and its associated energy would drain from the small scales, at least until the field reaches equipartition with the velocity. This is qualitatively consistent with the observed deficit in the magnetic energy from the large k region in Figures 1 and 2 for $f_h > f_{h,\,\mathrm{crit}}$, as compared to $f_h < f_{h,\,\mathrm{crit}}$.

The growth of the actual peak in magnetic energy at the forcing scale is also aided by the fact that a forced kinetic helicity reduces the nonlinear transfer term in the Navier-Stokes equation. The nonlinear term is $-\boldsymbol{v}\cdot\nabla\boldsymbol{v} = \boldsymbol{v}\times\boldsymbol{\omega} - \nabla v^2$. When helicity is present, the $\boldsymbol{v}\times\boldsymbol{\omega}$ term is reduced. For subsonic turbulence, the v^2 contribution to the evolution equation should be inconsequential. Thus, since the main cascade driver is reduced for helical turbulence, the kinetic energy requires more time to cascade, providing a bit more time for this energy to be transferred directly into stretching the magnetic field near the input scale. Although a cascade of magnetic energy steadily drains the field from the input scale, the hold up of the kinetic energy cascade means that there is more time to resupply the field to a larger amplitude before draining, compared to the $f_h < f_{h,\,\mathrm{crit}}$ case.

The total kinetic energy density for $k \geq k_f$ is fixed and is always larger than the total magnetic energy density for $k \geq k_f$. If $f_h < f_{h,\,\mathrm{crit}}$, then there is a significant nonhelical part of the field that feels no tendency to inverse cascade. This fraction piles up quickly on the small scales (Chou 2001; Maron & Cowley 2001). If this fraction dominates, then the spectrum will be dominated by the nonhelical turbulence dynamics. If $f_h > f_{h,\,\mathrm{crit}}$, the magnetic energy associated with the helicity, which inverse cascades, dominates. A lower limit on $f_{h,\,\mathrm{crit}}$ can be found from modeling the large-scale field growth as an α^2 dynamo that has a growth rate of $\alpha - \beta s_1$, where β is turbulent diffusion and $s_1 = 1$ is the growing large-scale wavenumber. But initially α and β have their kinematic values and $\alpha/s_1\beta \sim 2f_h s_f/(3s_1)$, and so initial growth requires at least $f_h > s_1/s_f \sim 0.33 = f_{\mathrm{crit}}$. This is roughly consistent with our results within small factors on the order of 1.

At the risk of applying our "slow" dynamo spectral shape results too cavalierly, we consider the implications for the Galaxy. The Galactic field has both a large-scale ($\gtrsim$2 kpc) and a small-scale component ($\lesssim$100 pc) (e.g., Zweibel & Heiles 1997). The small-scale field (which has a magnitude about a few times larger than the large-scale field) appears to have a peak at the forcing scale, as does the kinetic energy (Armstrong, Rickett, & Spangler 1995), and the two are in near equipartition with $v \sim b \sim 10$ km s^{-1}. Ignoring the "slow" versus "fast" issue for the moment, our results would imply that the Galactic magnetic spectrum (with its small-scale peak at the forcing scale) is consistent only with turbulence forced with $f_h > f_{h,\,\mathrm{crit}}$. Note that we have been considering only externally forced turbulence, as opposed to self-generated turbulence from shear. This particular assumption, at least, is consistent with the Galaxy, where supernovae are the primary driver (Sellwood & Balbus 1999). (Given that our box dynamo is "slow," we should note that well-motivated analytic interpretations for generation of "fast" large-scale magnetic energy in sheared rotators that appeal less explicitly to kinetic helicity, or not at all, have been studied [see Balbus & Hawley 1998; Vishniac & Cho 2001], as do specific proposals for how boundaries might enable fast helical α-Ω dynamos [Blackman 2002]. We do not discuss these further, but note that in the Galaxy, a net magnetic flux in the Galactic disk in addition to magnetic energy seems to be needed.)

Our results might also suggest that kinetic helicity plays a role in the protogalactic small-scale dynamo model (Kulsrud et al. 1997) of the large-scale Galactic field. In this model, the large-scale field of the Galaxy results from gravitational collapse and flux freezing of the small-scale protogalactic field. The model requires that small-scale dynamos generate significant power at the forcing scale, and our results would suggest this is possible only in $\mathrm{Pr} \geq 1$ plasmas when $f_h \geq f_{h,\,\mathrm{crit}}$.

5. CONCLUSIONS

The magnetic spectrum of MHD turbulence forced in a periodic box with fractional kinetic helicity above a critical value $f_{h,\,\mathrm{crit}}$ saturates with two peaks: a large-scale peak and a peak at the forcing scale, when $\mathrm{Pr} \geq 1$. If $f_h < f_{h,\,\mathrm{crit}}$, there is only one peak in the spectrum, and it is at the resistive scale. Although the turbulence is not strictly Alfvénic for any f_h, it is much more nearly Alfvénic for larger f_h. The range of scales, the boundary conditions, and the actual nature of helical forcing pose obstacles when comparing nonsheared periodic box simulations with real astrophysical rotators. Nevertheless, the important qualitative implication of our results is that helical forcing not only influences the large-scale magnetic field spectra of forced turbulent systems, but may also help account for observed peaks at the forcing scale.

E. G. B. acknowledges DOE grant DE-FG02-00ER54600. J. M. benefited from supercomputers at the National Center for Supercomputing Applications at the University of Illinois at Urbana-Champaign (1000 CPU hr) and from their very helpful staff. We thank S. Boldyrev, A. Brandenburg, S. Cowley, G. Field, J. McWilliams, V. Pariev, A. Schekochihin, and E. Vishniac for discussions.

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